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Learning buyers' valuation distribution in posted-price selling*

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Summary. A dynamic pricing model is studied where a seller of an asset faces a sequence of potential buyers whose valuation distribution is unknown to the seller. The seller learns more about the distribution in the selling process and becomes less optimistic as the object remains unsold. We characterize the optimal posted prices which incorporate updated beliefs every period, and derive a rather tight sufficient condition under which these prices decline over time. An example is provided where the optimal prices can actually increase over time if the condition is violated.

Keywords and Phrases: Price determination, Posted-price selling, Learning

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I. Introduction

How should a seller price an asset when facing a sequence of potential buyers whose valuations are drawn from an unknown distribution? When all buyers' valuations are random draws from a given distribution, it is well known that the optimal posted prices for the seller are time-invariant: the seller offers a fixed price to each sequentially arriving buyer (cf. Riley and Zeckhauser, 1983; McAfee and McMillan, 1988). However, in many economic situations, a seller may not know precisely the valuation distribution of the potential buyers. For example, a seller of an asset, such as a house or a firm, may not know the exact market conditions which affect the willingness to pay of all buyers in a similar way. As time goes by, the seller learns more about the distribution of buyers' willingness to pay, and thus adjusts the price accordingly. One purpose of this note, therefore, is to contribute to the theory of dynamic pricing by characterizing the optimal prices in a situation where learning in the selling process plays an important role in price determination.

We consider a model in which a seller with one indivisible object tries to sell it to potential buyers in countably many periods. Potential buyers arrive sequentially, one in each period. (Equivalently, we can think of a procurer approaching potential producers of a product sequentially.) The seller posts a price at each period, and if a buyer does not buy at the posted price, she departs within the period. We interpret this as the seller making a take-it-or-leave-it price offer to each buyer when she arrives.[1] All buyers' valuations are drawn from the same distribution, which the seller is uncertain about. In this note, this uncertainty is modelled simply by assuming that there are two possible valuation distributions. In the selling process, the seller obtains more information about each buyer's valuation and updates her belief about which distribution these valuations were drawn from.

Our main finding is that the sequence of optimal prices for the seller decline over time under the commonly used assumption that the hazard rate of one distribution is higher

[1] We are, thus, studying the optimal pricing problem within a specific selling mechanism. We focus on situations where selling formats that involve direct competition among buyers, such as auctions, are not practical. This may be due to the infrequent arrival or the impatience of buyers. It is discussed further in the concluding section.

than that of the other. When the true distribution is common knowledge, this condition captures the property that that buyers' valuations are uniformly higher in one distribution than in another, and it guarantees that the monopoly price with the former distribution is higher than that with the latter distribution. When the true distribution is unknown, as time passes, the seller becomes more convinced that the true distribution is the lower one and thus sets prices towards the monopoly price corresponding to the lower distribution. Thus the hazard-rate relation seems a natural condition to ensure that the monopoly seller's prices will monotonically decline over time.

In a model of learning, one might think that the optimal prices should *always* be declining as the seller becomes less optimistic. We provide a counter-example to this intuition. In this example, where the hazard-rate condition is violated but one distribution still first-order stochastically dominates the other, the optimal prices actually increase over time. This suggests that the hazard-rate condition we have obtained is not only natural, but also rather tight.

Our findings offer a possible explanation for the empirical observations that the price for an asset tends to decline as time passes (when it remains unsold), such as in the housing market. Such observations might appear to be at odds with the behavior of an optimizing monopoly seller (i.e., charging a constant price) as predicted by the existing theories, but they should be expected when the optimizing monopoly seller faces uncertainty about the true distribution of buyers' valuations.

In Section II, we set up the model and examine various properties of the optimal price path. Concluding remarks are offered in Section III.

II. The Model and the Analysis

A seller has a single object to sell. Potential buyers arrive sequentially, one in each period. Denote periods by t, $t = 1, 2, ...$ A potential buyer's valuation u is a random draw from either distribution $F(u)$ or distribution $G(u)$, both of which are continuous and differentiable on support $[\underline{u}, \bar{u}]$, with $0 \leq \underline{u} < \bar{u} < \infty$. The associated density functions are $f(u)$ and $g(u)$, respectively. Both $f(u)$ and $g(u)$ are strictly positive and differentiable on $[\underline{u}, \bar{u}]$. We assume that $F(u) < G(u)$ for $\underline{u} < u < \bar{u}$. Thus, $F(u)$ strictly first-order stochastically dominates $G(u)$. The *ex ante* probability that u is drawn from $F(u)$ is $\alpha_1 \in (0,1)$. The seller offers a take-it-or-leave-it price to the buyer in each period, until the good is sold. The seller's objective is to maximize the present value of the expected proceeds from the sale, which we shall call the seller's expected profit, with her discount factor being δ.

Let the probability that the seller assigns to distribution $F(u)$ be α_t at t. If the seller charges p_τ in period τ, $\tau = t, t+1, ...$, then her profit is given by

$$\begin{aligned}
&\pi(p_t, p_{t+1}, ...; \alpha_t) \\
= \;& \alpha_t \sum_{\tau=t}^{\infty} \prod_{k=t}^{\tau-1} F(p_k)[1-F(p_\tau)]p_\tau \delta^{\tau-t} + (1-\alpha_t) \sum_{\tau=t}^{\infty} \prod_{k=t}^{\tau-1} G(p_k)[1-G(p_\tau)]p_\tau \delta^{\tau-t} \\
= \;& [\alpha_t(1-F(p_t)) + (1-\alpha_t)(1-G(p_t))]p_t \\
& +\delta[\alpha_t F(p_t)(1-F(p_{t+1})) + (1-\alpha_t)G(p_t)(1-G(p_{t+1}))]p_{t+1} \\
& +\delta^2[\alpha_t F(p_t)F(p_{t+1})(1-F(p_{t+2})) + (1-\alpha_t)G(p_t)G(p_{t+1})(1-G(p_{t+2}))]p_{t+2} \\
& + ...,
\end{aligned}$$

where $\prod_{k=t}^{t-1} F(p_k) = 1$ by definition.

Therefore, the seller's continuation value (expected profit) of owning the object at the beginning of t with belief α_t is

$$\pi(\alpha_t) = \sup_{p_t, p_{t+1}, ...} \pi(p_t, p_{t+1}, ...; \alpha_t). \tag{1}$$

Without loss of generality, we can restrict all p_τ to the interval $[\underline{u}, \bar{u}]$. That is, $\underline{u} \le p_\tau \le \bar{u}$, $\forall \tau$. Since

$$\sum_{\tau=t}^{\infty} \prod_{k=t}^{\tau-1} F(p_k)[1-F(p_\tau)] = \sum_{\tau=t}^{\infty} \prod_{k=t}^{\tau-1} G(p_k)[1-G(p_\tau)] = 1,$$

and $\delta < 1$, it is easy to see that $\pi(p_t, p_{t+1}, ...; \alpha_t)$ is bounded by $\underline{u}$ and $\bar{u}$. Therefore, the $\pi(\cdot)$ defined by (1) must be finite.

The optimization problem defined by (1) is difficult to solve. Therefore, instead of solving (1) directly, we use the techniques of dynamic programming to analyze this problem. Given the seller's belief α_t in period t and the price p_t, her belief in period $t+1$ conditional on the good being unsold in period t is given by

$$\begin{aligned}\alpha_{t+1} &= \frac{\Pr(u < p_t \,|F) \Pr(F)}{\Pr(u < p_t \,|F) \Pr(F) + \Pr(u < p_t \,|G) \Pr(G)} \\ &= \frac{F(p_t)\alpha_t}{F(p_t)\alpha_t + G(p_t)(1-\alpha_t)} \equiv \alpha_{t+1}(\alpha_t, p_t). \end{aligned} \tag{2}$$

Using this updated belief, we can rewrite (1) as

$$\begin{aligned}\pi(\alpha_t) &= \max_{p_t} \{[\alpha_t(1 - F(p_t)) + (1-\alpha_t)(1 - G(p_t))]p_t \\ &\quad + \delta[\alpha_t F(p_t) + (1-\alpha_t)G(p_t)]\pi(\alpha_{t+1}(\alpha_t, p_t))\} \\ &\equiv \max_{p_t} \pi(p_t; \alpha_t) \end{aligned} \tag{3}$$

Any solution to (3) with the property that $\pi(\alpha) \leq \bar{u}$ for any $\alpha \in [0,1]$ must be the value function defined in (1). Furthermore, $\pi(\alpha)$ is continuously differentiable since $F(\cdot)$ and $G(\cdot)$ are continuously differentiable. (cf. Stokey and Lucas [1989; Theorem 4.11, p.85]) Clearly, $\pi(\alpha) > 0$ for any α. In what follows, we use both (1) and (3) when deriving the optimal price path.

Define

$$\begin{aligned} H_F(p_t, p_{t+1}, p_{t+2}, ...) &= p_t[1 - F(p_t)] + \delta F(p_t)[1 - F(p_{t+1})]p_{t+1} \\ &\quad + \delta^2 F(p_t)F(p_{t+1})[1 - F(p_{t+2})]p_{t+2} + ... \end{aligned}$$

and

$$\begin{aligned} H_G(p_t, p_{t+1}, p_{t+2}, ...) &= p_t[1 - G(p_t)] + \delta G(p_t)[1 - G(p_{t+1})]p_{t+1} \\ &\quad + \delta^2 G(p_t)G(p_{t+1})[1 - G(p_{t+2})]p_{t+2} + ... \end{aligned}$$

It is easy to see that $\bar{u} \geq H_F \geq \underline{u}$ and $\bar{u} \geq H_G \geq \underline{u}$.

Using the notation above, we can again rewrite (1) as

$$\pi(\alpha_t) = \sup_{p_t, p_{t+1}, \ldots} \{\alpha_t H_F(p_t, p_{t+1}, \ldots) + (1 - \alpha_t) H_G(p_t, p_{t+1}, \ldots)\} \tag{4}$$

Define

$$H_F^* = \sup_{p_t, p_{t+1}, \ldots} H_F(p_t, p_{t+1}, \ldots)$$

as the seller's optimal expected revenue when it is common knowledge that a buyer's valuation follows $F(\cdot)$. Similarly, define

$$H_G^* = \sup_{p_t, p_{t+1}, \ldots} H_G(p_t, p_{t+1}, \ldots)$$

as the seller's optimal expected revenue when it is common knowledge that a buyer's valuation follows $G(\cdot)$. Since H_F^* is obviously independent of t, we have

$$H_F^* = \max_p \{p[1 - F(p)] + \delta F(p) H_F^*\}.$$

Let p_F^* be the maximizer of H_F^*. It is then optimal for the seller to charge p_F^* each and every period if buyers' valuations are known to be from $F(\cdot)$. Thus,

$$H_F^* = H_F(p_F^*, p_F^*, \ldots) = \frac{p_F^*[1 - F(p_F^*)]}{1 - \delta F(p_F^*)}.$$

Similarly, Let p_G^* be the maximizer of H_G^*, we have

$$H_G^* = H_G(p_G^*, p_G^*, \ldots) = \frac{p_G^*[1 - G(p_G^*)]}{1 - \delta G(p_G^*)}.$$

We can now establish some useful lemmas.

Lemma 1 Assume that $f(\underline{u})\underline{u} < \frac{1}{1-\delta}$ and $g(\underline{u})\underline{u} < \frac{1}{1-\delta}$. Then $\underline{u} < p_t^* < \bar{u}$, $\forall\ t = 1, 2, \ldots$.

Proof First, note that $p_t^* \in [\underline{u}, \bar{u}]$. Next, at the optimal prices, we have

$$\begin{aligned}
\left.\frac{\partial \pi(p_t, p_{t+1,\ldots}; \alpha_t)}{\partial p_t}\right|_{p_i = p_i^*, i = t, t+1, \ldots} &= \alpha_t f(p_t^*) \left[\frac{1 - F(p_t^*)}{f(p_t^*)} - p_t^* + \delta H_F(p_{t+1}^*, p_{t+2}^*, \ldots)\right] \\
&+ (1 - \alpha_t) g(p_t^*) \left[\frac{1 - G(p_t^*)}{g(p_t^*)} - p_t^* + \delta H_G(p_{t+1}^*, p_{t+2}^*, \ldots)\right] \\
&= 0,
\end{aligned}$$

if p_t^* is interior. If $p_t^* = \underline{u}$, we would have

$$\begin{aligned}
&\left.\frac{\partial \pi(p_t, p_{t+1,...}; \alpha_t)}{\partial p_t}\right|_{p_i = p_i^*, i=t,t+1,...} \\
= \quad & \alpha_t f(\underline{u}) \left[\frac{1}{f(\underline{u})} - \underline{u} + \delta H_F(p_{t+1}^*, p_{t+2}^*, ...)\right] \\
& +(1-\alpha_t) g(\underline{u}) \left[\frac{1}{g(\underline{u})} - \underline{u} + \delta H_G(p_{t+1}^*, p_{t+2}^*, ...)\right] \\
> \quad & 0,
\end{aligned}$$

noting that $H_F \geq \underline{u}$ and $H_G \geq \underline{u}$. This implies that p_t^* is not optimal. A contradiction. Thus, $p_t^* > \underline{u}$.

Finally, $p_t^* < \bar{u}$. If not, $p_t^* = \bar{u}$. Then, $\alpha_{t+1} = \alpha_t$ and $\pi(\alpha_t) = \delta\pi(\alpha_t)$ from equations (2) and (3). But then $\pi(\alpha_t) = 0$, which cannot be true.□

The above lemma provides a sufficient condition for the optimal prices to be interior. The assumption in Lemma 1 requires the lower end of a buyer's valuation distribution or its density be sufficiently low, so that setting a price equal to $\underline{u}$ is never optimal. (Note that this assumption is always satisfied if $\underline{u} = 0$.) In the rest of the paper, we maintain this assumption to guarantee that interior solutions are always obtained.

Lemma 2 At the optimum, $\alpha_{t+1} < \alpha_t$, $\forall t$.

Proof First, notice that at the optimum $\underline{u} < p_t^* < \bar{u}$ for all t from Lemma 1. Therefore, $F(p_t^*) < G(p_t^*)$, and

$$\alpha_{t+1} = \frac{F(p_t^*)\alpha_t}{F(p_t^*)\alpha_t + G(p_t^*)(1-\alpha_t)} < \frac{\alpha_t}{\alpha_t + (1-\alpha_t)} = \alpha_t.$$

□

Thus, as one would expect, the seller becomes less and less optimistic about the buyers' valuation distribution as the good remains unsold over time.

If the seller charges the same price every period, then since $F(\cdot)$ strictly stochastically dominates $G(\cdot)$, one would expect that the seller can expect to earn more when the true distribution is $F(\cdot)$ than when it is $G(\cdot)$. This is confirmed in the following lemma.

Lemma 3 $H_F(p,p,...) > H_G(p,p,...),\ \forall\ p \in (\underline{u}, \bar{u})$.

Proof Since

$$H_F(p,p,...) = p[1-F(p)] + \delta F(p) H_F(p,p,...),$$

we have

$$H_F(p,p,...) = \frac{p[1-F(p)]}{1-\delta F(p)}.$$

Similarly,

$$H_G(p,p,...) = \frac{p[1-G(p)]}{1-\delta G(p)}.$$

Therefore,

$$\begin{aligned}
& H_F(p,p,...) - H_G(p,p,...) \\
= \ & \frac{p[1-F(p)][1-\delta G(p)] - p[1-G(p)][1-\delta F(p)]}{[1-\delta F(p)][1-\delta G(p)]} \\
= \ & \frac{(1-\delta)[G(p)-F(p)]}{[1-\delta F(p)][1-\delta G(p)]} > 0.
\end{aligned}$$

□

We are now ready to establish some properties for the seller's profit function and the optimal prices.

Proposition 1 $\pi'(\alpha_t) \geq 0,\ \forall \alpha_t \in [0,1)$.

Proof

$$\pi(\alpha_t) = \alpha_t H_F(p_t^*, p_{t+1}^*, ...) + (1-\alpha_t) H_G(p_t^*, p_{t+1}^*, ...).$$

So,

$$\pi'(\alpha_t) = H_F(p_t^*, p_{t+1}^*, ...) - H_G(p_t^*, p_{t+1}^*, ...)$$

by the envelope theorem.

Since $\pi(0) = H_G(p_G^*, p_G^*, ...) = H_G^*$, we have

$$\pi'(0) = H_F(p_G^*, p_G^*, ...) - H_G(p_G^*, p_G^*, ...) > 0$$

from Lemma 3.

We now claim that $\pi'(\alpha_t) \geq 0$ for all $\alpha_t \in [0,1)$. Suppose the claim is not true. Then there exists some $\tilde{\alpha} \in (0,1)$ such that $\pi'(\tilde{\alpha}) < 0$. Recall that $\pi'(\alpha)$ is continuous. From the properties of continuous functions, there must exist at least one $\alpha \in (0, \tilde{\alpha})$ such that $\pi'(\alpha) = 0$. Let $\widehat{\alpha} = \sup\{\alpha \in [0, \tilde{\alpha}] : \pi'(\alpha) = 0\}$. Since $\pi'(\alpha)$ is continuous, $\pi'(\widehat{\alpha}) = 0$. Furthermore, $\pi'(\alpha_t) < 0$ for $\alpha_t \in (\widehat{\alpha}, \tilde{\alpha}]$. Thus $\pi(\tilde{\alpha}) < \pi(\widehat{\alpha})$.

But since $\pi'(\widehat{\alpha}) = 0$, we have

$$H_F(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...) = H_G(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...).$$

Therefore,

$$\begin{aligned}
\pi(\widehat{\alpha}) &= \widehat{\alpha} H_F(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...) + (1-\widehat{\alpha}) H_G(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...) \\
&= \tilde{\alpha} H_F(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...) + (1-\tilde{\alpha}) H_G(p_t^*(\widehat{\alpha}), p_{t+1}^*(\widehat{\alpha}), ...) \\
&\leq \pi(\tilde{\alpha}),
\end{aligned}$$

where the inequality above is due to the fact that $p_1^*(\widehat{\alpha}), p_2^*(\widehat{\alpha}), ...$ may not be the optimal prices associated with $\tilde{\alpha}$. But then we have a contradiction.$\square$

The above proposition states that the seller's expected profit is an increasing function of her belief that the true distribution is $F(\cdot)$. From Lemma 2, this belief is decreasing over time. Therefore, the seller's expected profit declines as time passes.

Proposition 2 (The declining price property) Assume that $\frac{f(p)}{1-F(p)} < \frac{g(p)}{1-G(p)}$ for $p \in (\underline{u}, \bar{u})$. Then $p_t^* > p_{t+1}^*$, for $t = 1, 2, ...$

Proof First, note that $H_F(p_t^*, p_{t+1}^*, ...) \geq H_G(p_t^*, p_{t+1}^*, ...)$ from $\pi'(\alpha_t) \geq 0$ and the envelope theorem.

Next,

$$\left.\frac{\partial \pi(p_t, p_{t+1,...;}\alpha_t)}{\partial p_t}\right|_{p_i=p_i^*, i=t,t+1,...} = \alpha_t f(p_t^*) A_F + (1-\alpha_t) g(p_t^*) A_G = 0, \tag{5}$$

where

$$A_F = \frac{1-F(p_t^*)}{f(p_t^*)} - p_t^* + \delta H_F(p_{t+1}^*, p_{t+2}^*, ...),$$

and

$$A_G = \frac{1-G(p_t^*)}{g(p_t^*)} - p_t^* + \delta H_G(p_{t+1}^*, p_{t+2}^*, ...).$$

Since $H_F(p_{t+1}^*, p_{t+2}^*, ...) \geq H_G(p_{t+1}^*, p_{t+2}^*, ...)$ and $\frac{1-F(p_t^*)}{f(p_t^*)} > \frac{1-G(p_t^*)}{g(p_t^*)}$, we have $A_F > A_G$.

Since $\alpha_1 \in (0,1)$ by assumption, $p_t^* \in (\underline{u}, \bar{u})$ from Lemma 1, we have $\alpha_t \in (0,1)$ from Equation (2). Aslo, the density functions are strictly positive by assumption. Therefore $\alpha_t f(p_t^*) > 0$ and $(1-\alpha_t) g(p_t^*) > 0$ for any t. Equation (5) thus implies $A_F > 0$ and $A_G < 0$. Therefore,

$$\left.\frac{\partial}{\partial \alpha_t}\left(\frac{\partial \pi(p_t, p_{t+1}, ...; \alpha_t)}{\partial p_t}\right)\right|_{p_i=p_i^*, i=t,t+1,...} = f(p_t^*) A_F - g(p_t^*) A_G > 0.$$

On the other hand, at p_t^*, the first-order condition $\frac{\partial \pi(p_t^*, \alpha_t)}{\partial p_t} = 0$, must also be satisfied, where $\pi(p_{t;}\alpha_t)$ is defined in equation (3). Taking the total derivative of both sides of $\frac{\partial \pi(p_t^*, \alpha_t)}{\partial p_t} = 0$ with respect to α_t, we have

$$\frac{d}{d\alpha_t}\left(\frac{\partial \pi(p_t^*, \alpha_t)}{\partial p_t^*}\right) = \frac{\partial^2 \pi(p_t^*, \alpha_t)}{\partial p_t^{*2}} \frac{dp_t^*}{d\alpha_t} + \frac{\partial}{\partial \alpha_t}\left(\frac{\partial \pi(p_t^*, \alpha_t)}{\partial p_t^*}\right) = 0. \tag{6}$$

Because p_t^* is the optimal price, $\frac{\partial^2 \pi(p_t^*,\alpha_t)}{\partial p_t^{*2}} \leq 0$. But since

$$\frac{\partial}{\partial \alpha_t}\left(\frac{\partial \pi(p_t^*,\alpha_t)}{\partial p_t^*}\right) = \left.\frac{\partial^2 \pi(p_t, p_{t+1}, ...; \alpha_t)}{\partial \alpha_t \partial p_t}\right|_{p_i=p_i^*, i=t,t+1,...} > 0,$$

equation (6) above can hold only if $\frac{\partial^2 \pi(p_t^*,\alpha_t)}{\partial p_t^{*2}} < 0$. Thus,

$$\frac{dp_t^*}{d\alpha_t} = -\frac{\frac{\partial}{\partial \alpha_t}\left(\frac{\partial \pi(p_t^*,\alpha_t)}{\partial p_t^*}\right)}{\frac{\partial^2 \pi(p_t^*,\alpha_t)}{\partial p_t^{*2}}} > 0.$$

Since $\alpha_t > \alpha_{t+1}$, for $t = 1, 2, ...$, from Lemma 2, we must have $p_t^* > p_{t+1}^*$, for $t = 1, 2, ...\Box$

Given the condition on the hazard rates of $F(\cdot)$ and $G(\cdot)$, the prices will fall over time. This is a natural condition for the declining-price property since it guarantees that the optimal price is higher when the true distribution is known to be $F(\cdot)$ than when it is known to be $G(\cdot)$. As time passes, it is more likely that the distribution is $G(\cdot)$. Therefore, the prices are adjusted downward towards the monopoly price for $G(\cdot)$.

The hazard-rate condition is satisfied by many familiar distributions, such as $F(u) = u^2$, $G(u) = u$, $u \in [0,1]$. Note that $\frac{f(p)}{1-F(p)} < \frac{g(p)}{1-G(p)}$ implies $F(\cdot)$ first-order stochastically dominates $G(\cdot)$. This is because by integrating both sides of the above inequality, we have $-\ln[1-F(p)] < -\ln[1-G(p)]$, which implies that $F(p) < G(p)$. Nevertheless, the hazard-rate condition is a fairly tight condition; it cannot be relaxed to a condition of first-order stochastic domination. When the hazard-rate condition does not hold, the optimal prices with learning can surprisingly be increasing over time, even though one distribution still stochastically dominates the other, as the following example illustrates. For computational convenience, the example uses discrete distributions.

Example Let $F(u)$ be such that $u = 0$ with probability 0.1, $u = 1$ with probability 0.6, and $u = 2$ with probability 0.3, and let $G(u)$ be such that $u = 0$ with probability 0.7, $u = 1$ with probability 0.1, and $u = 2$ with probability 0.2. It is straightforward to check that $F(u)$ first-order stochastically dominates $G(u)$. Suppose that $\delta = 0.5$, and $\alpha_1 = 0.95$.

In this example, the only candidates for the optimal prices are $p = 1$ and $p = 2$. Simple calculations show that the seller should first set the optimal price at 1. If the good remains unsold, the seller becomes less optimistic about the true distribution being $F(u)$ and α_t continues to drop. When α_t drops below the critical level, 0.852, the seller's optimal price jumps up to 2, and remains at 2 thereafter.

The reason that optimal prices actually increase over time in this example is quite intuitive: When α is high, the seller should set the price to 1 since the optimal price would be 1 if the true distribution were known to be $F(u)$; and when α is low, the seller should set the price to 2 since the optimal price would be 2 if the true distribution were known to be $G(u)$. The discrete equivalence of the assumption in Proposition 2 is violated here, as that assumption implies that the optimal price would be lower when the distribution is known to be the stochastically dominated one.[2]

As a corollary of Proposition 2, we have

Corollary Assume that $\frac{f(p)}{1-F(p)} < \frac{g(p)}{1-G(p)}$ for $p \in (\underline{u}, \bar{u})$. Then $\alpha_t \to 0$ as $t \to \infty$.

Proof First, note that since p_t^* monotonically decreases as t increases and since $\underline{u} < p_t^* < \bar{u}$, $\underline{p} = \lim_{t\to\infty} p_t^*$ exists. Next, $\underline{p} > \underline{u}$. If not, then $\underline{p} = \underline{u}$. For any T,

$$\begin{aligned} 0 &= \frac{\partial \pi(p_T^*, p_{T+1}^*, ...; \alpha_T)}{\partial p_T} \\ &= \alpha_T f(p_T^*) \left[\frac{1 - F(p_T^*)}{f(p_T^*)} - p_T^* + \delta H_F(p_{T+1}^*, p_{T+2}^*, ...) \right] \\ &\quad + (1-\alpha_T) g(p_T^*) \left[\frac{1 - G(p_T^*)}{g(p_T^*)} - p_T^* + \delta H_G(p_{T+1}^*, p_{T+2}^*, ...) \right]. \end{aligned} \tag{7}$$

By assumption, as T goes to infinity, p_T^* goes to $\underline{u}$. Therefore, from the assumption in Lemma 1,

$$\lim_{t\to\infty} f(p_T^*) \left[\frac{1 - F(p_T^*)}{f(p_T^*)} - p_T^* + \delta H_F(p_{T+1}^*, p_{T+2}^*, ...) \right]$$

[2]Note that for continuous distributions on [0,2] that can be approximated by our discrete distributions, it is also possible that optimal prices will increase over time, at least for some periods.

$$\geq \quad f(\underline{u})\left[\frac{1}{f(\underline{u})}-\underline{u}+\delta\underline{u}\right]>0.$$

Similarly,

$$\lim_{t\to\infty} g(p_T^*)\left[\frac{1-G(p_T^*)}{g(p_T^*)}-p_T^*+\delta H_G(p_{T+1}^*,p_{T+2}^*,...)\right]$$

$$\geq \quad g(\underline{u})\left[\frac{1}{g(\underline{u})}-\underline{u}+\delta\underline{u}\right]>0.$$

Hence, the right-hand side of (7) does not converge to zero as T goes to infinity. We have a contradiction. Thus, $p_t^* \geq \underline{p} > 0$, $\forall t$.

Let

$$\beta=\min_t\left\{\frac{G(p_t^*)}{F(p_t^*)}\right\},$$

Since p_t^* decreases as t increases, $\underline{u} < \underline{p} \leq p_t^* \leq p_1^* < \bar{u}$ for all t. Therefore, we have $\beta > 1$.

Now,

$$\begin{aligned}\alpha_{t+1} &= \frac{\alpha_t F(p_t^*)}{\alpha_t F(p_t^*)+(1-\alpha_t)G(p_t^*)}\\ &= \frac{\alpha_t}{\alpha_t+(1-\alpha_t)\frac{G(p_t^*)}{F(p_t^*)}}\\ &\leq \frac{\alpha_t}{\alpha_t+(1-\alpha_t)\beta}\end{aligned}$$

But since $\alpha_t+(1-\alpha_t)\beta$ decreases in α_t when $\beta>1$, and $\alpha_1 \geq \alpha_t$, we have

$$\alpha_t+(1-\alpha_t)\beta \geq \alpha_1+(1-\alpha_1)\beta > 1.$$

Thus

$$\alpha_{t+1}\leq\frac{\alpha_t}{\alpha_1+(1-\alpha_1)\beta}=\left(\frac{1}{\alpha_1+(1-\alpha_1)\beta}\right)^t\alpha_1,$$

which goes to 0 as $t\to\infty$.□

Thus, if the good remains unsold for a sufficiently long time, the seller must believe that the true distribution is almost surely $G(u)$.

III. Concluding Remarks

The study of optimal pricing in dynamic settings has typically assumed that the distribution of buyers' valuations is known. The seller's optimal prices exhibit stationary properties in such situations. In this note, we have studied a model where the distribution of buyers' valuations is unknown but can be learned as the seller experiences rejections of her price offers by buyers. Under a very natural condition, the seller's optimal prices decline over time, as she becomes less optimistic about the possible distribution of buyers' valuations. This offers an explanation of why prices for an asset tend to be lower the longer it remains unsold, as is often observed in the housing market.[3]

While the optimal behavior of the seller necessarily involves learning in our model, it does not follow that the seller in our model will do better than a seller in a similar environment where learning is not possible. To see this, consider a situation that is the same as our model except where each buyer's valuation is an independent draw from a known distribution $\alpha_1 F(u) + (1-\alpha_1)G(u)$. Clearly, there can be no learning about the distribution in such a situation. Denote the seller's revenue by $\tilde{\pi}$ in this case. We have

$$\tilde{\pi} = \max_p \left\{ \frac{p[1-\alpha_1 F(p) - (1-\alpha_1)G(p)]}{1-\delta\alpha_1 F(p) - \delta(1-\alpha_1)G(p)} \right\}.$$

But since

$$\begin{aligned} \pi(\alpha_1) &= [\alpha_1(1-F(p_1^*)) + (1-\alpha_1)(1-G(p_1^*))]p_1^* + \\ &\quad \delta[\alpha_1 F(p_1^*) + (1-\alpha_1)G(p_1^*)]\pi(\alpha_2(\alpha_1, p_1^*)) \\ &\le [\alpha_1(1-F(p_1^*)) + (1-\alpha_1)(1-G(p_1^*))]p_1^* + \\ &\quad \delta[\alpha_1 F(p_1^*) + (1-\alpha_1)G(p_1^*)]\pi(\alpha_1), \end{aligned}$$

[3] There can be other reasons why prices decline over time, such as the seller is unable to commit to a fixed price. Our point here is that even if the seller has all the commitment power, the optimal prices may still decline due to learning.

we have

$$\begin{aligned}\pi(\alpha_1) &\leq \frac{p[1-\alpha_1 F(p_1^*) - (1-\alpha_1)G(p_1^*)]}{1-\delta\alpha_1 F(p_1^*) - \delta(1-\alpha_1)G(p_1^*)} \\ &\leq \max_p \left\{ \frac{p[1-\alpha_1 F(p) - (1-\alpha_1)G(p)]}{1-\delta\alpha_1 F(p) - \delta(1-\alpha_1)G(p)} \right\} = \tilde{\pi}.\end{aligned}$$

Thus, comparing our model with a model without information updating in the selling process, and supposing that the seller starts with the same prior belief about the buyers' valuation distribution in both cases, his expected profit will actually be higher when there is no new information being learned in the selling process.

Our model is closely related to the literature on optimal selling strategies, such as Chen and Rosenthal (1996), McAfee and McMillan (1988), and Wang (1993). The novel part of our model is to add uncertainty to the distribution of the buyers' valuations. Our assumption that the seller is uncertain about only two distributions is strong, and is made for tractability. We believe that our main finding, that prices decline overtime when the hazard rates of distributions can be ranked uniformly, will continue to hold when there are more than two distributions. Of course, the actual analysis would be much more complicated. Another restriction of our analysis is that we have set up our model as one of optimal pricing, rather than one of finding the optimal selling mechanism. We have chosen the less ambitious approach in this note because the dynamic optimal pricing problem with unknown buyers' valuation distribution is itself interesting and the time path of prices in such a model has not been studied before. Furthermore, posted-price selling is the most commonly seen selling format, and it is also likely to be an optimal mechanism when the arrival of buyers are infrequent or buyers are myopic. For future research, nevertheless, it is desirable to generalize our model to address the design of optimal selling mechanisms in environments involving learning.

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